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Models with $SL(2, C)$ symmetry and their S-matrices

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Abstract

We study quantum systems whose scattering modes are governed by unitary representation of the $\mathfrak{sl}(2, C)$ algebra. The S-matrices of the systems under consideration are defined from intertwining relations for the Weyl equivalent representations of the group $SL(2, C)$ or its Lie algebra.

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1. Introduction

The dynamical role of Lorentz group $SO(3, 1) \approx SL(2, C)$ is familiar since the seminal work of Fock [1] addressing the solution of the Coulomb problem. Following that, Bargmann [2] recognized that the angular momentum and Runge–Lenz vectors generate the Lie algebra of $SO(3, 1)$ in the subspace of positive energies. It was realized that the Hamiltonian H belongs to the centre of the enveloping algebra of $SO(3, 1)$, i.e., H is a function of the Casimir operator C of $SO(3, 1)$. Namely, $H \sim 1/(C - 1)$.

From this point of view the Coulomb scattering problem has been studied by many authors [3–6]. It was first shown by Zwanziger [5] that the algebra of $SO(3, 1)$ may be used to calculate the Coulomb phase shift. However, his method still made use of coordinate realization for the generators. Wu *et al* [6] building on the works [7–9] have suggested a purely algebraic method for the calculation of the Coulomb S-matrix.

As usual, in this reference the scattering operator is defined in terms of asymptotic states

$$\Psi_{\text{out}} = S'\Psi_{\text{in}}.$$

Here Ψ_{in} and Ψ_{out} are incoming and outgoing asymptotic states which are assumed to satisfy the free Schrödinger equation

$$H^0\Phi_\alpha = E\Phi_\alpha.$$

These states are described by the Euclidean group $E(3)$ in three dimensions. It is therefore mandatory to find an interrelation between dynamical algebra, which describes actual states, and the Euclidean algebra, which describes the freely evolving states. However, due to the absence of a general procedure for the description of such connection formulae, it is rather difficult to derive the S-matrix for problems with higher-rank groups.

A treatment of the scattering problem based on a different approach was given in [10]. It has been argued that the scattering operator S for models whose Hamiltonians belong to the centre of the enveloping algebra of some non-compact group G is constrained to satisfy

$$SU(g) = \tilde{U}(g)S \qquad \text{for all} \quad g \in G \tag{1}$$

or

$$S\,\mathrm{d}U(a) = \mathrm{d}\tilde{U}(a)S \qquad \text{for all} \quad a \in \mathfrak{g} \tag{2}$$

where U and $\tilde{U}$ are the Weyl equivalent representations of G, while $\mathrm{d}U$ and $\mathrm{d}\tilde{U}$ are the corresponding representations of the algebra $\mathfrak{g}$ of G. (The representations U and $\tilde{U}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.)

It should be emphasized that the scattering operator S in equations (1) and (2) is the so-called modified scattering operator [8], i.e.,

$$\Psi_\alpha^{(+)} = S\Psi_\alpha^{(-)} \tag{3}$$

where $\Psi_\alpha^{(+)}$ and $\Psi_\alpha^{(-)}$ are the actual states satisfying the time-independent Schrödinger equation

$$H\Psi_\alpha^{(\pm)} = E\Psi_\alpha^{(\pm)}.$$

In other words we use the scattering operator S defined in terms of actual states rather than the scattering operator S' defined in terms of asymptotic states. For systems under consideration, it is more appropriate to use S because its matrix elements are evaluated with respect to actual states

$$S_{\beta\alpha} = \left(\Psi_\beta^{(+)}, S\Psi_\alpha^{(+)}\right) = \left(\Psi_\beta^{(-)}, S\Psi_\alpha^{(-)}\right) = (\Phi_\beta, S'\Phi_\alpha). \tag{4}$$

The principal difference between S and S' can be easily seen from the representations

$$S = \Omega^{(+)}\Omega^{(-)\dagger}, \qquad S' = \Omega^{(-)\dagger}\Omega^{(+)} \tag{5}$$

where $\Omega^{(+)}$ and $\Omega^{(-)}$ are the Møller operators mapping the whole Hilbert space $\mathcal{H}$ onto the subspace $\mathcal{H}^c$ of scattering states

$$\Psi_\alpha^{(\pm)} = \Omega^{(\pm)}\Phi_\alpha. \tag{6}$$

This means that S is defined and unitary in $\mathcal{H}^c$, while S' is defined and unitary in the whole $\mathcal{H}$. Moreover

$$[S, H] = 0, \qquad [S', H^0] = 0.$$

Thus, one can in principle evaluate the S-matrix (more precisely, the submatrices belonging to the definite value of energy) from (1) or (2) without ever writing the wavefunction, or the relation between the actual states and the asymptotic one, or the interrelation between the original algebra $\mathfrak{g}$ and the 'free' algebra, or ever mentioning the concepts of space and time.

At this stage we note that operator A is said to intertwine the representations U and $\tilde{U}$ of the group G if the relation $AU(g) = \tilde{U}(g)A$ holds [12]. It turns out that [10, 13–17] if the scattering system is governed by a Hamiltonian describable as

$$H = f(C)|_{\mathcal{H}} \tag{7}$$

or

$$Q(H - E) = (C - q)|_{\mathcal{H}} \tag{8}$$

the scattering operator for such a system is also related to the intertwining operator, but now

$$S = A|_{\mathcal{H}} \tag{9}$$

where $\mathcal{H}$ is a subspace occurring in the subgroup reductions, q is an eigenvalue of C and Q is some nontrivial operator.

This paper is the first of two devoted to the study of scattering problems related to $SL(2, C)$ by the algebraic method proposed in [10]. It is the purpose of this work to re-examine and re-derive all three-dimensional scattering problems [1, 18, 19, 21] known to possess the Lorentz group as a dynamical group. In this discussion we shall restrict ourselves to the non-relativistic Coulomb scattering with and without spin, and the dyon–dyon scattering, although the case of Bogomolny–Prasad–Sommerfeld monopole scattering [20, 21] can also be easily treated. It is shown that the scattering amplitude for all these problems can be derived from equation (B34). The phase of the amplitude, which shows up in interference phenomena, has been given explicitly. Our other purpose here is to prepare all the necessary background for the second paper in the series, devoted to two-dimensional and one-dimensional problems.

2. The Coulomb problem

The hydrogen atom was one of the first quantum systems with which the importance of the algebraic approach was realized. The model was investigated in various aspects in both the Heisenberg and the Schrödinger formulation of quantum mechanics and the 'accidental' degeneracy was understood naturally in terms of the underlying Lie algebraic structure.

Let us start the discussion with the fact that for the Coulomb system, governed by the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2M} + \frac{\alpha}{r} \tag{10}$$

the angular momentum $\mathbf{L}$ and the Runge–Lenz vector $\mathbf{A}$, which are given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \qquad \mathbf{A} = \frac{1}{2M}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) + \alpha \hat{\mathbf{r}} \tag{11}$$

are conserved

$$[\mathbf{L}, H] = 0, \qquad [\mathbf{A}, H] = 0 \tag{12}$$

where, M is the (reduced) mass, $\hat{\mathbf{r}} = \mathbf{r}/r$ and α denotes the strength of the potential, which for the hydrogen atom is equal to $-e^2$. From now on we will use the system of units in which $M = \hbar = 1$. The components of $\mathbf{L}$ and $\mathbf{A}$ have the following commutation relations:

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk} L_k, \qquad [L_i, A_j] = \mathrm{i}\epsilon_{ijk} A_k, \qquad [A_i, A_j] = -\mathrm{i}\epsilon_{ijk} L_k (2H). \tag{13}$$

Moreover the following relations hold:

$$\mathbf{L} \cdot \mathbf{A} = 0, \qquad \mathbf{A}^2 = 2H(\mathbf{L}^2 + 1) + \alpha^2. \tag{14}$$

Due to (12), we may restrict algebra (13) to the subspace where H has a definite value and write

$$\mathbf{K} = \mathbf{A}(2H)^{-1/2} \tag{15}$$

since we are concerned only with the positive spectrum of H, i.e., $H = p^2/2$, $p > 0$. In this way, we obtain the Lie algebra of the Lorentz group $SO(3, 1) \sim SL(2, C)$ spanned by the six (Hermitian) operators L_i, K_i. Then, the S-matrix for the Coulomb problem can be defined from equation (1) or (2). To this end, a few facts from representation theory of the group $SL(2, C)$ are useful [22].

The Lie algebra of $SL(2, C)$ is spanned by the six elements J_i, N_i, $i = 1, 2, 3$, obeying the following commutation relations:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k, \qquad [J_i, N_j] = \mathrm{i}\epsilon_{ijk} N_k, \qquad [N_i, N_j] = -\mathrm{i}\epsilon_{ijk} J_k. \tag{16}$$

The J_i span the Lie algebra of $SU(2) \sim SO(3)$. The unitary irreducible representations (UIRs) of $SL(2, C)$ are known to form two series: principal and complementary. It is also known that only the principal series of the UIRs of $SL(2, C)$ describe the scattering states. Consequently, the relevant unitary representations will be the principal series.

The principal series of $SL(2, C)$ are characterized by the pair $\chi = (\rho, \nu)$, where $\nu = 0, \pm\frac{1}{2}, \pm 1, \ldots$, while $-\infty < \rho < \infty$. The representations specified by labels $\chi = (\rho, \nu)$ and $\tilde{\chi} = (\rho', \nu')$ are Weyl equivalent if and only if $\rho' = -\rho$ and $\nu' = -\nu$. In every UIR of principal series of $SL(2, C)$ the Casimir invariants C_1 and C_2

$$C_1 = \mathbf{J}^2 - \mathbf{N}^2, \qquad C_2 = \mathbf{J} \cdot \mathbf{N} \tag{17}$$

become equal to a multiple of the identity operator

$$C_1 = -\nu^2 - \rho^2 - 1, \qquad C_2 = \nu\rho. \tag{18}$$

The operators $\mathbf{J}^2$ and J_3 may be diagonalized simultaneously with C_1 and C_2. Thus we may introduce a basis in the following way:

$$\mathbf{J}^2|\rho\nu; jm\rangle = j(j+1)|\rho\nu; jm\rangle, \qquad J_3|\rho\nu; jm\rangle = m|\rho\nu; jm\rangle \tag{19}$$

where

$$j = |\nu|, |\nu| + 1, |\nu| + 2, \ldots, \qquad m = j, j-1, \ldots, -j.$$

Once the algebra has been established we may consider realizations of the generators L_i, K_i different from those given above. We note that equation (14) restricts us to the most degenerate principal series representations with

$$\rho = \alpha/p, \qquad \nu = 0. \tag{20}$$

These representations can be realized in the Hilbert space spanned by the eigenvectors $|lm\rangle$ of $\mathbf{L}^2$ and L_3. The operators L_i, K_i are then defined by (see appendix A)

$$\begin{aligned}
&L_3|lm\rangle = m|lm\rangle \\
&L_\pm|lm\rangle = [(l \mp m)(l \pm m + 1)]^{\frac{1}{2}}|l, m \pm 1\rangle \\
&K_3|lm\rangle = \mathrm{i}(-1 + \mathrm{i}\rho - l)a_{l+1,m}|l+1, m\rangle + \mathrm{i}(\mathrm{i}\rho + l)a_{l,m}|l-1, m\rangle \\
&K_\pm|lm\rangle = \pm\mathrm{i}(1 - \mathrm{i}\rho + l)b_{l+1,\pm m+1}|l+1, m \pm 1\rangle \pm \mathrm{i}(\mathrm{i}\rho + l)b_{l,\mp m}|l-1, m \pm 1\rangle
\end{aligned}$$

where $L_\pm = L_1 \pm \mathrm{i}L_2$, $K_\pm = K_1 \pm \mathrm{i}K_2$, $|lm\rangle \equiv |\rho 0; lm\rangle$ and

$$a_{l,m} = \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}, \qquad b_{l,m} = \sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}}. \tag{21}$$

We are now prepared to compute the S-matrix. To do this let us write equation (2) for generators L_3, $L_\pm$ and K_3

$$SL_3 = \tilde{L}_3 S \tag{22}$$

$$SL_\pm = \tilde{L}_\pm S \tag{23}$$

$$SK_3 = \tilde{K}_3 S. \tag{24}$$

Applying both sides of equations (22) and (23) to the basis vector $|lm\rangle$ we find that the S-matrix in the angular-momentum representation is diagonal and its matrix elements are independent of m, i.e.,

$$S|lm\rangle = S_l|lm\rangle. \tag{25}$$

The value of its diagonal elements can be defined by the use of (24). As a result we obtain the recurrence relation

$$(1 - \mathrm{i}\rho + l)S_{l+1} = (1 + \mathrm{i}\rho + l)S_l, \qquad (-\mathrm{i}\rho + l)S_l = (\mathrm{i}\rho + l)S_{l-1} \tag{26}$$

which implies that

$$S_l = \varkappa \frac{\Gamma(1 + \mathrm{i}\rho + l)}{\Gamma(1 - \mathrm{i}\rho + l)} \tag{27}$$

where $\varkappa$ is a phase factor. (Since S is unitary, each of its eigenvalues has modulus 1.) The factor $\varkappa$ is constant with respect to l but may depend on ρ. This question will be discussed later on. (The problem is closely related to the study of analytic properties of the scattering amplitude.) It follows that for the Coulomb potential one can put $\varkappa(\rho) = 1$.

Thus, the S-matrix on the energy shell (i.e. between two states of the same energy) is given by

$$\langle l'm'|S|lm\rangle = \delta_{m'm}\delta_{l'l}S_l.$$

In other words

$$\langle E'l'm'|S|Elm\rangle = \delta_{m'm}\delta_{l'l}\delta(E' - E)S_l. \tag{28}$$

Once the S-matrix has been obtained in the angular-momentum representation we may transform it to one defined in the momentum representation. Taking into account that the transformation function is given by [23]

$$\langle \mathbf{p}|Elm\rangle = (p)^{-\frac{1}{2}}\delta(E_p - E)Y_l^m(\mathbf{n}), \qquad E_p = \frac{p^2}{2}, \quad \mathbf{n} = \mathbf{p}/p \tag{29}$$

where Y_l^m are the spherical harmonics, we find

$$\begin{aligned}\langle \mathbf{p}'|S|\mathbf{p}\rangle &= \frac{1}{4\pi p}\delta(E_{p'} - E_p)\sum_{l=0}^{\infty}(2l + 1)S_l P_l(\mathbf{n}\cdot\mathbf{n}') \\ &= \frac{2^{\mathrm{i}\rho-1}}{\pi p}\delta(E' - E)\frac{\Gamma(1 + \mathrm{i}\rho)}{\Gamma(-\mathrm{i}\rho)}(1 - \mathbf{n}\cdot\mathbf{n}')^{-1-\mathrm{i}\rho}.\end{aligned} \tag{30}$$

For the scattering amplitude $f(E_p, \theta)$ defined by

$$\langle \mathbf{p}'|S - 1|\mathbf{p}\rangle = \frac{\mathrm{i}}{2\pi}\delta(E_{p'} - E_p)f(E_p, \theta) \tag{31}$$

this gives

$$f(E_p, \theta) = \frac{1}{2\mathrm{i}p}\frac{\Gamma(1 + \mathrm{i}\rho)}{\Gamma(-\mathrm{i}\rho)}\frac{1}{\sin^2\frac{\theta}{2}}\exp\left[-\mathrm{i}\rho\ln\left(\sin^2\frac{\theta}{2}\right)\right], \qquad \theta \neq 0 \tag{32}$$

with $\rho = \alpha/p$ and $\cos\theta = \mathbf{n}\cdot\mathbf{n}'$.

The result concerning the scattering amplitude can also be obtained by means of the following arguments. Let $\mathcal{L}_2(S^2)$ denote the Hilbert space of square-integrable functions $\phi(\mathbf{n})$, $\mathbf{n} \in S^2$, on the two-dimensional sphere S^2. The representation $(\rho, 0)$ of $SL(2, C)$ can be realized on $\mathcal{L}_2(S^2)$ [24]

$$(U(g)\phi)(\mathbf{n}) = (\omega)^{-1+\sigma}\phi(\mathbf{n}_g) \tag{33}$$

where $\sigma = \mathrm{i}\rho$, while ω, $\mathbf{n}_g$ are defined from

$$g^{-1}\zeta = \omega\zeta_g \tag{34}$$

where $\zeta \equiv (\mathbf{n}, 1)$ and $\zeta_g \equiv (\mathbf{n}_g, 1)$ are null vectors of the Minkowski space $R^{3.1}$. (For the explicit expression of $U(g)$, see appendix A.) In this realization the operator S is defined as

$$(S\phi)(\mathbf{n}) = \int K(\mathbf{n}, \mathbf{n}')\phi(\mathbf{n}')\,\mathrm{d}\mathbf{n}' \tag{35}$$

where $\mathrm{d}\mathbf{n}$ is the invariant measure on S^2. Thus, equation (1) will serve to fix the dependence of the kernel $K(\mathbf{n}, \mathbf{n}')$ on $\mathbf{n}$ and $\mathbf{n}'$. Equality (1) implies that

$$(SU(g)\phi)(\mathbf{n}) = (\widetilde{U}(g)S\phi)(\mathbf{n}). \tag{36}$$

Hence, the kernel K is constrained to satisfy

$$K(\mathbf{n}_g, \mathbf{n}'_g) = (\omega)^{1+\sigma}(\omega')^{1+\sigma}K(\mathbf{n}, \mathbf{n}'). \tag{37}$$

In deriving equation (37) we have used the relation

$$\mathrm{d}\mathbf{n}_g = (\omega)^{-2}\,\mathrm{d}\mathbf{n}. \tag{38}$$

The kernel K is, up to a constant, uniquely determined from (37) and is given by

$$K(\mathbf{n}, \mathbf{n}') = c(1 - \mathbf{n}\cdot\mathbf{n}')^{-1-\sigma}. \tag{39}$$

The verification of equation (39) is based on the relation

$$(1 - \mathbf{n}_g\cdot\mathbf{n}'_g) = (\omega)^{-1}(\omega')^{-1}(1 - \mathbf{n}\cdot\mathbf{n}') \tag{40}$$

which is obviously a consequence of the relation

$$[g^{-1}\zeta, g^{-1}\zeta'] = [\zeta, \zeta']$$

where $[\zeta, \zeta'] \equiv \zeta_1\zeta'_1 + \zeta_2\zeta'_2 + \zeta_3\zeta'_3 - \zeta_4\zeta'_4$ is the Minkowskian scalar product.

Hence, the improper matrix elements $\langle E'\mathbf{n}'|S|E\mathbf{n}\rangle$ are given by

$$\langle E'\mathbf{n}'|S|E\mathbf{n}\rangle = c\delta(E' - E)(1 - \mathbf{n}\cdot\mathbf{n}')^{-1-\sigma}. \tag{41}$$

The coefficient c is determined to within a phase factor by the unitarity condition

$$\int \langle E''\mathbf{n}''|S|E\mathbf{n}\rangle\overline{\langle E'\mathbf{n}'|S|E\mathbf{n}\rangle}\,\mathrm{d}E\,\mathrm{d}\mathbf{n} = \delta(E'' - E')\delta(\mathbf{n}''-\mathbf{n}')$$

which gives

$$|c|^2 = \frac{|\sigma|^2}{4\pi^2}$$

(the bar denotes complex conjugate).

In order to fix the phase of c (or, equivalently, $\varkappa$), we need to know the analytic properties of the S-matrix. It follows from (35) that the operator S is well defined and analytic in σ for $\mathrm{Re}\,\sigma < 0$. To see this, one can compute the S-matrix in the angular-momentum representation. From (35) we have (see appendix B)

$$\langle l'm'|S|lm\rangle = \pi c\delta_{m'm}\delta_{l'l}2^{1-\sigma}\frac{\Gamma(-\sigma)}{\Gamma(1+\sigma)}\frac{\Gamma(1+\sigma+l)}{\Gamma(1-\sigma+l)}, \qquad \mathrm{Re}\,\sigma < 0.$$

However, the integral (35) can be continued analytically in σ to give a meromorphic function defined in the entire complex plane σ. To this end, we put

$$c(\sigma) = \frac{2^{\sigma-1}}{\pi}\frac{\Gamma(1+\sigma)}{\Gamma(-\sigma)}.$$

Observe that for an attractive Coulomb potential ($\alpha < 0$) it produces poles which correspond to bound states. Moreover, with this factor the operator S becomes unitary for $\sigma = \mathrm{i}\rho$.

Finally, taking into account that $|\mathbf{p}\rangle = (p)^{-1/2}|E\mathbf{n}\rangle$, with $E = p^2/2$, $\mathbf{n} = \mathbf{p}/p$, we find that

$$\langle \mathbf{p}'|S|\mathbf{p}\rangle = \frac{2^{\mathrm{i}\rho-1}}{\pi p}\delta(E' - E)\frac{\Gamma(1+\mathrm{i}\rho)}{\Gamma(-\mathrm{i}\rho)}(1 - \mathbf{n}\cdot\mathbf{n}')^{-1-\mathrm{i}\rho}. \tag{42}$$

In conclusion, we note that an interrelation between the coordinate realization and realization (33) of representation $(\rho, 0)$ is given by

$$F(\mathbf{r}) = \int \kappa(\mathbf{r}, \mathbf{n})\phi(\mathbf{n})\,\mathrm{d}\mathbf{n} \tag{43}$$

with

$$\kappa(\mathbf{r}, \mathbf{n}) = \exp(\mathrm{i}\alpha r/\rho)\,{}_1F_1\left[1 + \mathrm{i}\rho; 1; -\frac{\mathrm{i}\alpha}{\rho}(r - \mathbf{r}\cdot\mathbf{n})\right] \tag{44}$$

where ${}_1F_1$ is a confluent hypergeometric function [25]. The action of generators (11) of $SL(2, C)$ on the function $F(\mathbf{r})$ induces via (43) a corresponding action of $SL(2, C)$ on $\phi(\mathbf{n})$ given by (A10).

3. The Coulomb scattering with spin

It is natural to extend the ordinary Coulomb problem to the Coulomb problem with spin thereby extending the most degenerate representation of the $\mathfrak{sl}(2, C)$ algebra to a non-degenerate one. Such a generalization was recently discussed by Levay and Amos [18].

The generalization of algebra (13) which incorporates a spin operator $\mathbf{S}$ is given by [18]

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} + \mathbf{S} \tag{45}$$

$$\mathbf{A}' = \tfrac{1}{2}(\mathbf{D}\times\mathbf{J} - \mathbf{J}\times\mathbf{D}) + \alpha\hat{\mathbf{r}} \tag{46}$$

where

$$\mathbf{D} = \mathbf{p} + \frac{1}{r^2}(\mathbf{S}\times\mathbf{r}) \tag{47}$$

One may verify that the operators $\mathbf{J}$ and $\mathbf{A}'$ satisfy the commutation relations

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, A'_j] = \mathrm{i}\epsilon_{ijk}A'_k, \qquad [A'_i, A'_j] = -\mathrm{i}\epsilon_{ijk}J_k(2H'). \tag{48}$$

Here H' is the modified Coulomb Hamiltonian

$$H' = \tfrac{1}{2}\mathbf{D}^2 + \frac{\alpha}{r} + \frac{1}{2r^2}\Lambda \tag{49}$$

$$= -\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)^2 + \frac{1}{2r^2}\mathbf{J}^2 + \frac{\alpha}{r} \tag{50}$$

where $\Lambda = (\mathbf{S}\cdot\hat{\mathbf{r}})$. The Hamiltonian H' commutes with $\mathbf{J}$ and $\mathbf{A}'$

$$[H', \mathbf{J}] = [H', \mathbf{A}'] = 0. \tag{51}$$

Hence the operators $\mathbf{J}$ and $\mathbf{A}'$ define conserved quantities. Moreover, the restriction $\mathbf{J}$ and $\mathbf{A}'$ to the eigenspace of H' corresponding to the eigenvalue $p^2/2$ $(p > 0)$ will lead to the Lie algebra (16) with

$$\mathbf{N} = \frac{1}{p}\mathbf{A}'.$$

If we compute the Casimir operators for this realization of $\mathfrak{sl}(2, C)$, they become a function of Λ

$$C_1 = \Lambda^2 - \left(\frac{\alpha}{p}\right)^2 - 1, \qquad C_2 = \frac{\alpha}{p}\Lambda. \tag{52}$$

We have therefore a reducible representation of $\mathfrak{sl}(2, C)$ acting in the eigenspace of H'.

We can express the representation of $\mathfrak{sl}(2, C)$ generated by (45) and (46) as a direct sum of UIRs of the principal series. Indeed, since Λ is a scalar operator, it commutes with each component of the total angular momentum $\mathbf{J}$. Consequently,

$$[H', \Lambda] = 0.$$

This means that we can decompose the carrier space into eigenspaces of the operator Λ and that the eigenspace with a definite value for Λ will carry a single UIR of the principal series of $\mathfrak{sl}(2, C)$ with

$$\rho = \frac{\alpha}{p}, \qquad \nu = \lambda. \tag{53}$$

Here λ is the eigenvalue of Λ which takes the values $\lambda = -s, \ldots, s$, where s is the spin of the particle. Since Λ is the component of the spin in the direction of motion of the particle one can interpret the eigenvalue λ as a helicity.

Thus for every pair of numbers p, s we obtain a $2s + 1$ irreducible representation of the $\mathfrak{sl}(2, C)$ algebra. This is due to the spin term in (45). (In the scalar case we have only one irreducible representation of $\mathfrak{sl}(2, C)$.)

In appendix B, we calculate the S-matrix in the angular-momentum basis. As a result, we have

$$\langle E'\lambda' j'm'|S|E\lambda jm\rangle = \delta(E' - E)\delta_{\lambda',-\lambda}\delta_{j'j}\delta_{m'm}S_j \tag{54}$$

where

$$S_j = \frac{\Gamma(1 + j + \mathrm{i}\rho)}{\Gamma(1 + j - \mathrm{i}\rho)} \tag{55}$$

Hence, the scattering amplitude for the case that the initial momentum is along the z-axis has the form [23]

$$f_{\lambda'\lambda}(E, \theta, \varphi) = \delta_{\lambda',-\lambda}\frac{1}{4\pi p}\sum_{j=|\lambda|}^{\infty}(2j + 1)(S_j - 1)\overline{D^j_{\lambda,-\lambda}(\varphi, \theta, -\varphi)} \tag{56}$$

where (θ, φ) specify the direction of the final momentum and D^j is the rotation matrix as defined in [26]

$$D^j_{mm'}(\varphi, \theta, \varphi') = \mathrm{e}^{-\mathrm{i}(m\varphi+m'\varphi')}d^j_{mm'}(\theta).$$

Comparing equations (56) and (B26) we obtain

$$f_{\lambda'\lambda}(E, \theta, \varphi) = \delta_{\lambda',-\lambda}\frac{1}{2\mathrm{i}p}\frac{\Gamma(1 + |\lambda| + \mathrm{i}\rho)}{\Gamma(|\lambda| - \mathrm{i}\rho)}\frac{1}{\sin^2\frac{\theta}{2}}\exp\left[-\mathrm{i}\rho\ln\left(\sin^2\frac{\theta}{2}\right) + 2\mathrm{i}\lambda\varphi + \mathrm{i}\pi(|\lambda| + \lambda)\right] \tag{57}$$

where $\rho = \alpha/p$. Hence, the unpolarized cross section (i.e. the cross section obtained by averaging over the initial helicities and summing over the final helicities) is found to be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{16E^2\sin^4\frac{\theta}{2}}\left[1 + \frac{2}{3}s(s + 1)\frac{E}{\alpha^2}\right]. \tag{58}$$

This result coincides with that [18] obtained by Zwanziger's method. Finally, we note that result (57) can also be obtained from (B34) by arguments very similar to those used in arriving at (42). We shall discuss this procedure for the dyon–dyon scattering.

4. Dyon–dyon scattering

In this section we study the scattering problem of two particles carrying both electric and magnetic charge (dyons). According to Zwanziger [19] the relative motion of such a system is governed by the Hamiltonian

$$H'' = \frac{1}{2}(\mathbf{p} - \mu\mathbf{D})^2 - \frac{\alpha}{r} + \frac{\mu^2}{2r^2} \tag{59}$$

with a vector potential $\mathbf{D}(\mathbf{r})$ given by

$$\mathbf{D}(\mathbf{r}) = \frac{(\mathbf{m}\cdot\mathbf{r})\mathbf{r}\times\mathbf{m}}{r[r^2 - (\mathbf{r}\cdot\mathbf{m})^2]} \tag{60}$$

where the unit vector $\mathbf{m}$ determines the direction of the string propagated from $-\infty$ to ∞ and

$$\alpha = -(e_1e_2 + g_1g_2)/4\pi, \qquad \mu = (e_1g_2 - g_1e_2)/4\pi \tag{61}$$

where e_i and g_i are the electric and magnetic charges.

Due to the rotational symmetry of the problem, we have a conserved quantity, a total angular momentum, which is given by

$$\mathbf{J} = \mathbf{r}\times(\mathbf{p} - \mu\mathbf{D})^2 - \mu\hat{\mathbf{r}}. \tag{62}$$

There is further a conserved vector analogous to the Runge–Lenz vector of the Coulomb problem

$$\mathbf{A}'' = \tfrac{1}{2}[(\mathbf{p} - \mu\mathbf{D})\times\mathbf{J} - \mathbf{J}\times(\mathbf{p} - \mu\mathbf{D})] - \alpha\hat{\mathbf{r}}. \tag{63}$$

So that

$$[\mathbf{J}, H''] = 0, \qquad [\mathbf{A}'', H''] = 0. \tag{64}$$

(Observe that operators (62) and (63) reduce to those of the hydrogen atom, in the special case $\mu = 0$.) Moreover, we have

$$\begin{aligned} &[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k && [J_i, A''_j] = \mathrm{i}\epsilon_{ijk}A''_k \\ &[A''_i, A''_j] = -\mathrm{i}\epsilon_{ijk}J_k(2H'') && \mathbf{J}\cdot\mathbf{A}'' = \alpha\mu \\ &\mathbf{A}''^2 - (\alpha)^2 = 2H''(\mathbf{J}^2 - \mu^2 + 1). \end{aligned} \tag{65}$$

As in the case of the Coulomb problem, we consider the subspace corresponding to the definite value $p^2/2$, $(p > 0)$ of H'' and define a new operator by

$$\mathbf{N} = \frac{1}{(2H'')^{1/2}}\mathbf{A}'' \tag{66}$$

and in this way $\mathbf{J}$ and $\mathbf{N}$ satisfy (16). For the Casimir operators C_1 and C_2, we have

$$C_1 = \mu^2 - \left(\frac{\alpha}{p}\right)^2 - 1, \qquad C_2 = \frac{\alpha}{p}\mu. \tag{67}$$

We have therefore the principal series representation of $\mathfrak{sl}(2, C)$ with

$$\rho = \alpha/p, \qquad \nu = \mu \tag{68}$$

acting in the eigenspace of H''.

After this is done, it is almost straightforward to get an explicit form of the scattering amplitude for the dyon–dyon scattering. Taking into account that

$$\langle\mathbf{p}'|S|\mathbf{p}\rangle = \frac{1}{4\pi p}\delta(E' - E)\langle\mathbf{n}'|S|\mathbf{n}\rangle$$

with $\mathbf{n} = \mathbf{p}/p$ and $\mathbf{n}' = \mathbf{p}'/p'$, $p' = p$, we find from equation (B34)

$$\langle \mathbf{p}'|S|\mathbf{p}\rangle = \mathrm{i}^{2(|\mu|-\mu)}\delta(E'-E)\frac{2^{-1+\mathrm{i}\frac{\alpha}{p}}}{\pi p}\frac{\Gamma\left(1+|\mu|+\mathrm{i}\frac{\alpha}{p}\right)}{\Gamma\left(|\mu|-\mathrm{i}\frac{\alpha}{p}\right)}(1-\mathbf{n}\cdot\mathbf{n}')^{-1-\mathrm{i}\frac{\alpha}{p}}\,\mathrm{e}^{2\mathrm{i}\mu\beta} \tag{69}$$

where $\beta \equiv \beta(\mathbf{n}, \mathbf{n}')$ is given by

$$\mathrm{e}^{-\mathrm{i}\beta(\mathbf{n},\mathbf{n}')} = \frac{\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\,\mathrm{e}^{-\mathrm{i}\varphi} - \sin\frac{\theta'}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi'}}{\left|\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\,\mathrm{e}^{-\mathrm{i}\varphi} - \sin\frac{\theta'}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi'}\right|}. \tag{70}$$

For the scattering amplitude $f(E; \mathbf{n}, \mathbf{n}')$ defined by

$$\langle \mathbf{p}'|S-1|\mathbf{p}\rangle = \frac{\mathrm{i}}{2\pi}\delta(E'-E)f(E; \mathbf{n}, \mathbf{n}')$$

this gives

$$f(E; \mathbf{n}, \mathbf{n}') = \mathrm{i}^{2(|\mu|-\mu)}\frac{2^{\mathrm{i}\frac{\alpha}{p}}}{\mathrm{i}p}\frac{\Gamma\left(1+|\mu|+\mathrm{i}\frac{\alpha}{p}\right)}{\Gamma\left(|\mu|-\mathrm{i}\frac{\alpha}{p}\right)}(1-\mathbf{n}\cdot\mathbf{n}')^{-1-\mathrm{i}\frac{\alpha}{p}}\,\mathrm{e}^{2\mathrm{i}\mu\beta(\mathbf{n},\mathbf{n}')} \tag{71}$$

where $\mathbf{n}' \neq \mathbf{n}$. (Observe that the scattering amplitude (71) reduces to those of the hydrogen atom, in the case $\mu = 0$.) If we choose the initial momentum $\mathbf{p}$ along the z-axis (and then use θ, φ for the direction of $\mathbf{p}'$), we obtain

$$f(E, \theta, \varphi) = \frac{\mathrm{i}^{2(|\mu|+\mu)}}{2\mathrm{i}p}\frac{\Gamma\left(1+|\mu|+\mathrm{i}\frac{\alpha}{p}\right)}{\Gamma\left(|\mu|-\mathrm{i}\frac{\alpha}{p}\right)}\left(\sin^2\frac{\theta}{2}\right)^{-1-\mathrm{i}\frac{\alpha}{p}}\exp(2\mathrm{i}\mu\varphi), \qquad \theta \neq 0 \tag{72}$$

(cf equation (57)). It is also worth noting that

$$f(E, \theta, \varphi) = \frac{1}{4\pi p}\sum_{j=|\mu|}^{\infty}(2j+1)\frac{\Gamma(1+j+\mathrm{i}\rho)}{\Gamma(1+j-\mathrm{i}\rho)}\overline{D^{j}_{\mu,-\mu}(\varphi, \theta, -\varphi)}, \qquad \theta \neq 0 \tag{73}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(E, \theta, \varphi)|^2 = \frac{\alpha^2 + 2\mu E}{4E^2(1-\cos\theta)^2}. \tag{74}$$

5. Conclusion

In this paper we explore the implications of Lorentz invariance for non-relativistic scattering processes whose Hamiltonians are related to the Casimir operators of $SL(2, C)$. It has been shown that the scattering problem can be completely solved within the framework of symmetry algebra, without explicit knowledge of a potential. Although in this paper we consider only scattering processes related to $SL(2, C)$, the intertwining operator seems to be the most adequate language to derive the scattering amplitude for systems with underlying dynamical groups.

Appendix A. The principal most degenerate series of $\mathfrak{so}(3, 1)$ algebra

To be able to use equation (2) in the computation of the S matrix, we have to know an abstract realization of the principal series of $\mathfrak{so}(3, 1) \simeq \mathfrak{sl}(2, C)$ algebra. For this reason we give here a realization of the principal series representation of $\mathfrak{so}(3, 1)$ algebra in the angular-momentum basis. For the sake of simplicity, we shall restrict ourselves to the most degenerate series representation.

We can obtain the most degenerate series representation of $\mathfrak{so}(3, 1)$ algebra from the corresponding representation of the $SO(3, 1)$ group. To do this, let us consider the most degenerate principal series representation of the $SO(3, 1)$ group realized in the Hilbert space $\mathcal{L}_2(S^2)$ of square-integrable function $\phi(\mathbf{n})$, $\mathbf{n} \in S^2$, on the two-dimensional unit sphere S^2. The representation operator $U(g)$, $g \in SO(3, 1)$ is defined by [24]

$$U(g)\phi(\mathbf{n}) = \left[\sum_{k=1}^{3}(g^{-1})_{4k}\, n_k + (g^{-1})_{44}\right]^{-1+\sigma} \phi(\mathbf{n}_g) \tag{A1}$$

where $\sigma = \mathrm{i}\rho$ and

$$(n_g)_i = \frac{\sum_{k=1}^{3}(g^{-1})_{ik}\, n_k + (g^{-1})_{i4}}{\sum_{k=1}^{3}(g^{-1})_{4k}\, n_k + (g^{-1})_{44}}, \qquad i = 1, 2, 3 \tag{A2}$$

with

$$n_1 = \sin\theta\cos\varphi, \qquad n_2 = \sin\theta\sin\varphi, \qquad n_3 = \cos\theta. \tag{A3}$$

Let us denote $\{g_{ij}(t)\}$, $i < j$, $i, j = 1, 2, 3, 4$ the one-parameter subgroups of $SO(3, 1)$ consisting of rotations or pseudorotations in $x_i - x_j$ planes, that is, of transformations of the form

$$x'_k = x_k, \qquad k \neq i, j, \qquad x'_i = x_i \cos t - x_j \sin t, \qquad x'_j = x_i \sin t + x_j \cos t \tag{A4}$$

or

$$x'_k = x_k, \qquad k \neq i, j, \qquad x'_i = x_i \cosh t + x_j \sinh t, \qquad x'_j = x_i \sinh t + x_j \cosh t. \tag{A5}$$

The (Hermitian) infinitesimal operators I_{ij} of the representation $U(g)$ corresponding to the one-parameter subgroups $\{g_{ij}(t)\}$ are defined by

$$I_{ij} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(g_{ij}(t))\bigg|_{t=0}. \tag{A6}$$

They are related to L_i, K_i as follows:

$$\mathbf{L} = (L_1, L_2, L_3) = (I_{23}, I_{31}, I_{12}) \tag{A7}$$

$$\mathbf{K} = (K_1, K_2, K_3) = (I_{14}, I_{24}, I_{34}). \tag{A8}$$

Instead of L_i, N_i it is more convenient to use their linear combinations

$$L_\pm = L_1 \pm \mathrm{i}L_2, \qquad K_\pm = K_1 \pm \mathrm{i}K_2. \tag{A9}$$

Then we have

$$\begin{aligned}
L_3 &= \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}\\
L_\pm &= \mathrm{e}^{\pm\mathrm{i}\varphi}\left(\pm\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi}\right)\\
K_3 &= \mathrm{i}(-1+\sigma)\cos\theta - \mathrm{i}\sin\theta\frac{\partial}{\partial\theta}\\
K_\pm &= \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\varphi}\left[(-1+\sigma)\cos\theta + \cos\theta\frac{\partial}{\partial\theta} \pm \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right].
\end{aligned} \tag{A10}$$

We take as a basis of the carrier space the eigenvector $|lm\rangle$ of $\mathbf{L}^2$ and L_3

$$\mathbf{L}^2|lm\rangle = l(l+1)|lm\rangle \qquad L_3|lm\rangle = m|lm\rangle.$$

It is not difficult to see that $|lm\rangle = Y_l^m$, where Y_l^m are the well-known spherical harmonics. Using the standard method, namely, applying the left-hand side and right-hand side of (A10) to the basis vectors, gives

$$
\begin{aligned}
&L_3|lm\rangle = m|lm\rangle \\
&L_\pm|lm\rangle = [(l \mp m)(l \pm m + 1)]^{1/2}|l, m \pm 1\rangle \\
&K_3|lm\rangle = \mathrm{i}(-1 + \sigma - l)a_{l+1,m}|l + 1, m\rangle + \mathrm{i}(\sigma + l)a_{l,m}|l - 1, m\rangle \\
&K_\pm|lm\rangle = \pm\mathrm{i}(1 - \sigma + l)b_{l+1,\pm m+1}|l + 1, m \pm 1\rangle \pm \mathrm{i}(\sigma + l)b_{l,\mp m}|l - 1, m \pm 1\rangle
\end{aligned}
$$

where

$$
a_{l,m} = \sqrt{\frac{(l + m)(l - m)}{(2l + 1)(2l - 1)}}, \qquad b_{l,m} = \sqrt{\frac{(l + m)(l + m - 1)}{(2l + 1)(2l - 1)}}.
$$

Appendix B. Evaluation of the *S*-matrix in an angular-momentum basis

In this section we calculate the S-matrices for scattering systems governed by principal non-degenerate series of representations of $SL(2, C)$. We find it expedient to use, for this purpose, equation (1). By realizing the principal series of $SL(2, C)$ on suitable Hilbert spaces of some functions we can derive from equation (1) the functional relations for the kernel of S which allow us to obtain an integral representation for the S-matrix. Thus, we can calculate the S-matrix in a straightforward manner from its integral formula.

One possibility of realizing the principal series of representations of $SL(2, C)$ is on the Hilbert space of square-integrable functions $f(z)$ of complex variables z. In this realization the representations of $SL(2, C)$ are given by [22]

$$
(U(g)f)(z) = \alpha(zg)f(z_g) \tag{B1}
$$

where $\alpha(g) = |g_{22}|^{-2\nu+2\sigma-2}(g_{22})^{2\nu}$, $\sigma = \mathrm{i}\rho$ and z_g is defined from

$$
\zeta g = k\zeta_g \tag{B2}
$$

where

$$
\zeta = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \qquad k = \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix}, \qquad \zeta_g = \begin{pmatrix} 1 & 0 \\ z_g & 1 \end{pmatrix}. \tag{B3}
$$

We may rewrite equation (B1) in the form

$$
(U(g)f)(z) = |g_{12}z + g_{22}|^{-2\nu+2\sigma-2}(g_{12}z + g_{22})^{2\nu} f\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right). \tag{B4}
$$

The operator S is defined as

$$
(Sf)(z) = \int Q(z, w)f(w)\,\mathrm{d}w. \tag{B5}
$$

Thus, equation (1) will serve to fix the dependence of the kernel $Q(z, w)$ on z and w. Equality (1) implies that

$$
(SU(g)f)(z) = (\widetilde{U}(g)Sf)(z). \tag{B6}
$$

So, the kernel Q is constrained to satisfy the functional equation

$$
\begin{aligned}
&\int Q(z, w)|g_{12}w + g_{22}|^{-2\nu+2\sigma-2}(g_{12}w + g_{22})^{2\nu} f\left(\frac{g_{11}w + g_{21}}{g_{12}w + g_{22}}\right)\mathrm{d}w \\
&\qquad = |g_{12}z + g_{22}|^{2\nu-2\sigma-2}(g_{12}z + g_{22})^{-2\nu} \int Q\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}, w'\right) f(w')\,\mathrm{d}w'
\end{aligned}
$$

Making the substitution

$$z = \frac{g_{22}z' - g_{21}}{-g_{12}z' + g_{11}}, \qquad w = \frac{g_{22}w' - g_{21}}{-g_{12}w' + g_{11}} \tag{B7}$$

and taking into account the formula

$$\mathrm{d}w = |-g_{12}w' + g_{11}|^{-4}\,\mathrm{d}w' \tag{B8}$$

we get the condition

$$Q\left(\frac{g_{22}z - g_{21}}{-g_{12}z + g_{11}}, \frac{g_{22}w - g_{21}}{-g_{12}w + g_{11}}\right) = |-g_{12}z + g_{11}|^{-2\nu+2\sigma-2}(-g_{12}z + g_{11})^{2\nu}$$
$$\times |-g_{12}z + g_{11}|^{-2\nu+2\sigma-2}(-g_{12}z + g_{11})^{2\nu} Q(z, w). \tag{B9}$$

In particular, choosing

$$g = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{B10}$$

in equation (B9), we obtain

$$Q(z - a, w - a) = Q(z, w). \tag{B11}$$

Hence, the kernel $Q(z, w)$ is a function of $z - w$ only

$$Q(z, w) = Q'(z - w) \tag{B12}$$

where $Q'(z) \equiv Q(z, 0)$. Taking into account this expression in (B9) we get

$$Q'\left(\frac{z}{g_{11}(-g_{12}z + g_{11})}\right) = |-g_{12}z + g_{11}|^{-2\nu+2\sigma-2}(-g_{12}z + g_{11})^{2\nu}|g_{11}|^{-2\nu+2\sigma-2}(g_{11})^{2\nu} Q'(z) \tag{B13}$$

where we have put $w = 0$. Choosing

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{B14}$$

in equation (B13), we see the $Q'(z)$ is an homogeneous function of z:

$$Q'\left(\frac{z}{a^2}\right) = |a|^{-4\nu+4\sigma+4} a^{4\nu} Q'(z). \tag{B15}$$

Then from (B15) it follows that

$$Q'(z) = \eta|z|^{2\nu-2\sigma-2}(z)^{-2\nu} \tag{B16}$$

where $\eta = Q'(1)$ is an arbitrary constant. Hence

$$Q(z, w) = c|z - w|^{2\nu-2\sigma-2}(z - w)^{-2\nu}. \tag{B17}$$

We mention that the operators S satisfying equation (1) are called intertwining operators between representations U and $\widetilde{U}$. These operators were first used to study the irreducibility of the principal series and the unitarity of the analytically continued representations (the complementary series) of $SL(2, C)$ [27]. Moreover, there is a general expression of the intertwining operators for induced representations of semisimple Lie groups [28]. Therefore, the above result can also be extracted from that expression. It is also worth noting that the integral in equation (B5) actually converges only for $\mathrm{Re}\,\sigma < 0$. (To see this, one can compute the S-matrix.) However, this integral may be continued analytically in σ to give meromorphic functions defined in the entire complex plane (see below).

For our purpose, however, it is more convenient to work in the compact picture [22], where the principal series of $SL(2, C)$ is realized on the Hilbert space of square-integrable functions ϕ on the group $SU(2)$ which obeys the condition

$$\phi(\gamma u) = \mathrm{e}^{\mathrm{i}\nu\varphi}\phi(u) \tag{B18}$$

where $u \in SU(2)$ and

$$\gamma = \begin{pmatrix} \exp(-\mathrm{i}\varphi/2) & 0 \\ 0 & \exp(\mathrm{i}\varphi/2). \end{pmatrix} \tag{B19}$$

In this realization the operators $U(g)$ are given by

$$U(g)\phi(u) = \frac{\alpha(ug)}{\alpha(u_g)}\phi(u_g) \tag{B20}$$

where u_g is defined from

$$ug = ku_g. \tag{B21}$$

The connection between these two realizations is given by

$$f(z) = \frac{1}{\sqrt{\pi}}\alpha^{-1}(u)\phi(u), \qquad z = \frac{u_{21}}{u_{22}}. \tag{B22}$$

Therefore, when the carries space of the representations is $L^2_\nu(SU(2))$ the operator S has the form

$$(S\phi)(u) = \int R(u, u')\phi(u')\,\mathrm{d}u' \tag{B23}$$

with the kernel given by

$$R(u, u') = \pi\eta|(uu'^{-1})_{21}|^{2\nu-2\sigma-2}[(uu'^{-1})_{21}]^{-2\nu}. \tag{B24}$$

Taking into account the fact that the basis states $|\rho\nu; jm\rangle$ (see equation (19)) in this realization differ from the matrix elements $t^j_{\nu m}$ [24] of the UIR of $SU(2)$ only by the factor $\sqrt{2j+1}$, we arrive at the integral formula for the S-matrix

$$\langle j'm'|S|jm\rangle = \sqrt{(2j+1)(2j'+1)}\int R(u, u')t^j_{\nu m}(u')\overline{t^j_{-\nu m'}(u)}\,\mathrm{d}u\,\mathrm{d}u'. \tag{B25}$$

If we introduce $u'u^{-1}$ instead of u' as a new variable we find that

$$\langle j'm'|S|jm\rangle = \delta_{jj'}\delta_{mm'}\int R'(u)t^j_{\nu,-\nu}(u)\,\mathrm{d}u \tag{B26}$$

with

$$R'(u) = \pi\eta|u_{21}|^{2\nu-2\sigma-2}[u_{21}]^{-2\nu}. \tag{B27}$$

On expressing the matrix u through the Euler angles ψ, θ, φ

$$u(\psi, \theta, \varphi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & \\ & \mathrm{e}^{-\mathrm{i}\psi/2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \mathrm{i}\sin\frac{\theta}{2} \\ \mathrm{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} & \\ & \mathrm{e}^{-\mathrm{i}\varphi/2} \end{pmatrix} \tag{B28}$$

with

$$\mathrm{d}u = \frac{1}{16\pi^2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\psi\,\mathrm{d}\varphi, \qquad 0 \leqslant \varphi < 2\pi, \qquad 0 \leqslant \theta < \pi, \qquad 0 \leqslant \psi < 4\pi \tag{B29}$$

and taking into account that [24]

$$t^j_{mm'}(u) = \mathrm{e}^{-\mathrm{i}(m\psi+m'\varphi)}P^j_{mm'}(\cos\theta) \tag{B30}$$

we have (see equation (7.512) (2) of [29])

$$\langle j'm'|S|jm\rangle = \pi\eta\delta_{jj'}\delta_{mm'}\frac{\Gamma(1+j+\sigma)}{\Gamma(1+j-\sigma)}\begin{cases}\frac{\Gamma(\nu-\sigma)}{\Gamma(1+\nu+\sigma)}, & \nu > 0\\ (-1)^{2\nu}\frac{\Gamma(-\nu-\sigma)}{\Gamma(1-\nu+\sigma)}, & \nu < 0\end{cases} \tag{B31}$$

for $\operatorname{Re}\sigma < 0$. The matrix elements for $\operatorname{Re}\sigma \geqslant 0$ are obtained by analytical continuation. For the scattering systems under consideration we put

$$\eta = \frac{\mathrm{i}^{2(|\nu|-\nu)}}{\pi}\frac{\Gamma(1+|\nu|+\sigma)}{\Gamma(|\nu|-\sigma)}. \tag{B32}$$

(Observe that η contains the bound-state poles of the amplitude.) With this factor the operator S becomes unitary for $\sigma = \mathrm{i}\rho$. This fact can also be visualized from equation (B31)

$$\langle j'm'|S|jm\rangle = \delta_{jj'}\delta_{mm'}\frac{\Gamma(1+j+\mathrm{i}\rho)}{\Gamma(1+j-\mathrm{i}\rho)} \tag{B33}$$

Due to equation (B18), the principal non-degenerate series of representations of $SL(2, C)$ can also be realized on the Hilbert space of functions $\phi(\mathbf{n}) \equiv \phi(u(\varphi, \theta, -\varphi))$ on $SU(2)/U(1) \simeq S^2$. In this realization

$$(S\phi)(\mathbf{n}) = c\int(1-\mathbf{n}\cdot\mathbf{n}')^{-1-\mathrm{i}\rho}\,\mathrm{e}^{2\mathrm{i}\nu\beta}\phi(\mathbf{n}')\,\mathrm{d}\mathbf{n}' \tag{B34}$$

where $\beta \equiv \beta(\mathbf{n}, \mathbf{n}')$ is given by

$$\mathrm{e}^{-\mathrm{i}\beta(\mathbf{n},\mathbf{n}')} = \frac{\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\,\mathrm{e}^{-\mathrm{i}\varphi} - \sin\frac{\theta'}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi'}}{\left|\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\,\mathrm{e}^{-\mathrm{i}\varphi} - \sin\frac{\theta'}{2}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi'}\right|}$$

and

$$c = \mathrm{i}^{2(|\nu|-\nu)}\frac{2^{\mathrm{i}\rho-1}}{\pi}\frac{\Gamma(1+|\nu|+\mathrm{i}\rho)}{\Gamma(|\nu|-\mathrm{i}\rho)}.$$

At the end, we mention the relation between $P^j_{mn}(\cos\theta)$ [24] and $d^j_{mn}(\cos\theta)$ [26]

$$P^j_{mn}(\cos\theta) = \mathrm{i}^{m-n}d^j_{mn}(\cos\theta). \tag{B35}$$

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